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On the Generalized Divisor Problem

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§1. The Generalized Divisor Problem

Let $d_z(n)$ be a multiplicative function defined by

$$\zeta^z(s) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \quad (\sigma > 1)$$

where $s = \sigma + it$, z is a complex number, and $\zeta(s)$ is the Riemann zeta function. Here $\zeta^z(s) = \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$.

We note that if z is a natural number, $d_z(n)$ coincides with the divisor function appearing in the Dirichlet-Piltz divisor problem, and $d_{-1}(n)$ with the Möbius function.

The generalized divisor problem is concerned with finding an asymptotic formula for $\sum_{n \leq x} d_z(n)$, which was observed for real $z > 0$ by A. Kienast [8] and K. Iseki [5] independently. A. Selberg [17] considered for all complex z , his result being

$$D_z(x) \equiv \sum_{n \leq x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O(x(\log x)^{\Re z - 2})$$

uniformly for $|z| \leq A$, $x \geq 2$, where A is any fixed positive number.

This was extended by Rieger [16] to arithmetic progressions such that

$$\begin{aligned} D_z(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} d_z(n) &= \left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\Gamma(z)\varphi(q)} (\log x)^{z-1} \\ &+ O\left(\left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\varphi(q)} (\log x)^{\Re z - 2} \log \log 4q\right) \end{aligned}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$, where A and τ are any fixed positive numbers.

On the other hand R.D. Dixon [4] derived the following formula which is a sharpening of Selberg's result;

$$D_z(x) \equiv \sum_{n \leq x} d_z(n) \\ = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z)}{(\log x)^m \Gamma(z-m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$, where $B_m(z)$ are regular functions of z , especially $B_0(z) = 1$.

We shall consider the connections between the asymptotic formula of $D_z(x, q, l)$ and the location of zeros of the Dirichlet L -function (Riemann zeta function). The main terms of above formulas are, however, inconvenient for our aim so that we introduce the following integral as the main term of $D_z(x, q, l)$:

$$\Phi_z(x, q) = \frac{1}{2\pi i} \int_L (L(s, \chi_0))^z \frac{x^s}{s} ds$$

where L is, for any τ ($0 < \tau < 1/2$), the path which begins at $1/2$, moves to $1 - \tau$ along the real axis, encircle the point 1 with radius τ in the counterclockwise direction, and returns to $1/2$ along the real axis. Here we denote by χ_0 the principal character *mod* q .

We note that if $z = k \in N$,

$$\Phi_k(x, 1) = x P_{k-1}(\log x), \quad \Phi_{-k}(x, q) = 0,$$

where $P_{k-1}(x)$ is the same polynomial of degree $k-1$ as in the Dirichlet divisor problem.

The error term is defined by

$$\Delta_z(x, q, l) = D_z(x, q, l) - \frac{1}{\varphi(q)} \Phi_z(x, q),$$

and let

$$\Theta(\chi) = \sup\{\sigma : L(\sigma + it, \chi) = 0\}, \quad \Theta_q = \max_{\chi \pmod{q}} \Theta(\chi).$$

THEOREM 1. *There exists some constant c such that*

$$\Delta_z(x, q, l) \ll x e^{-c\sqrt{\log x}}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$ where A and τ are any fixed positive numbers.

Further we have

$$\Delta_z(x, q, l) \ll x^{\Theta_q + \epsilon}$$

uniformly for $|z| \leq A$, $q \leq x$, $(q, l) = 1$.

Conversely if $\Delta_z(x, q, l) \ll x^{\Xi + \epsilon}$ for any l ($(q, l) = 1$) and for some $z \in C - Q^+$, where Q^+ denotes the set of all non negative rational numbers, then any $L(s, \chi) \pmod{q}$ has no zeros for $\sigma > \Xi$.

The main term $\Phi_z(x, q)$ has an asymptotic expansion

$$\Phi_z(x, q) = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z-m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$. Here N is any fixed positive integer and $B_m(z, q)$ ($0 \leq m \leq N-1$) are regular functions of z , especially $B_0(z, q) = (\varphi(q)/q)^z$.

Remark

1. For $q = 1$, we define α_z by

$$\alpha_z = \inf\{\alpha : \Delta_z(x, 1, 1) \ll x^\alpha\}$$

in a similar way as in the Dirichlet divisor problem. Then it is well known that

$$L. H. \iff \alpha_k \leq 1/2 \quad \text{for } \forall k \in N.$$

Theorem 1 shows that

$$R. H. \implies \alpha_z \leq 1/2 \quad \text{for } \forall z \in C,$$

Conversely,

$$\alpha_z \leq 1/2 \quad \text{for } \exists z \in C - Q^+ \implies R. H.$$

If we suppose that all the zeros of $\zeta(s)$ are simple, the last statement holds for $\exists z \in C - N$.

2. R. Balasubramanian and K. Ramachandra [1] have observed the following asymptotic formula by a method similar to that of Selberg [18] or H. Delange [2][3].

$$\sum_{nd(n) \leq x} 1 = M(x) + O(x \exp(-c(\log x)^{3/5}(\log \log x)^{-1/5}))$$

$$M(x) = \frac{1}{2\pi i} \int_C f(s) \frac{x^s}{s} ds \asymp \frac{x}{\sqrt{\log x}}.$$

where C is a suitable path which is similar to L .

3. A. Ivić [7] evaluated the integral $\int_1^T |\zeta(1+it)|^{2z} dt$ by using Theorem 1, where $z > 0$ is an arbitrary fixed real number.

4. Similar results are hold for the following sums :

$$\sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} z^{\omega(n)}, \quad \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} z^{\Omega(n)},$$

where $\omega(n)$ means the number of distinct prime factors of n , and $\Omega(n)$ means the number of total prime factors allowing multiplicity.

§2. The Asymptotic Formula for $\pi_k(x, q, l)$

Let $\pi_k(x)$ be the number of integers $\leq x$ which are products of k distinct primes. For $k = 1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x . C.F. Gauss stated empirically that $\pi_2(x) \sim x(\log \log x)/\log x$, and, by using the prime number theorem, E. Landau proved that $\pi_k(x) \sim x(\log \log x)^{k-1}/(k-1)!\log x$. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x) = \frac{xQ(\log \log x)}{\log x} + O\left(\frac{x(\log \log x)^k}{k!(\log x)^2}\right)$$

uniformly for $1 \leq k \leq A \log \log x$, where $Q(x)$ is polynomial of degree $k-1$.

H. Delange [3] obtained as a sharpening of Selberg's result,

$$\pi_k(x) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_m(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every $k \geq 1$, where N is any fixed integer ≥ 1 and $Q_m(x)$ are polynomials of degree not exceeding $k-1$.

We define $\pi_k(x, q, l)$ as a generalization of $\pi(x, q, l)$ by

$$\pi_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ n = p_1 \cdots p_k \ (p_i \neq p_j)}} 1.$$

In order to consider the connections between $\pi_k(x, q, l)$ and zeros of $L(s, \chi)$ ($\zeta(s)$) we define the following integral as the main term of $\pi_k(x, q, l)$:

$$F_{k, \delta}(x, q) = \frac{1}{(2\pi i)^2} \int_{|z|=1} \int_{L_\delta} (L(s, \chi_0))^z \times \left\{ \prod_p \left(1 + \frac{z\chi_0(p)}{p^s}\right) \left(1 - \frac{\chi_0(p)}{p^s}\right)^z \right\} \frac{1}{z^{k+1}} \frac{x^s}{s} ds dz$$

where L_δ is, for every δ and any r ($\delta > 0$, $r > 0$, $\delta + r < 1/2$), the path which begins at $1/2 + \delta$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2 + \delta$ along the real axis.

For $k = 1$, we can express this in terms of the logarithmic integral. Namely,

$$F_{1,\delta}(x, q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}).$$

The generating function of $\pi_k(x)$ is

$$\sum_{\ell=0}^k \frac{1}{\ell!(k-\ell)!} (\log \zeta(s))^\ell f^{(k-\ell)}(s, 0)$$

where

$$f(s, z) = \prod_p \left(1 + \frac{z}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^z,$$

$f^{(n)}(s, z)$ means the n -th derivative of $f(s, z)$ with respect to z .

The error term is defined by

$$R_{k,\delta}(x, q, l) = \pi_k(x, q, l) - \frac{1}{\varphi(q)} F_{k,\delta}(x, q).$$

THEOREM 2. *There is some constant c such that*

$$R_{k,\delta}(x, q, l) \ll x e^{-c\sqrt{\log x}}$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q, l) = 1$.

Further we have

$$R_{k,\delta}(x, q, l) \ll x^{\Theta_s + \epsilon}$$

uniformly for $k \geq 1$, $q \leq x$, $(q, l) = 1$.

Conversely if $R_{k,\delta}(x, q, l) \ll x^{\Xi + \epsilon}$ for any l ($(q, l) = 1$) and for some $k \geq 1$, then any $L(s, \chi) \pmod{q}$ has no zeros for $\sigma > \Xi$.

The main term $F_{k,\delta}(x, q)$ has an asymptotic expansion

$$F_{k,\delta}(x, q) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_{m,q}(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every k and q . Here N is any fixed positive integer and $Q_{m,q}(x)$ are polynomials of degree not exceeding $k-1$, especially the coefficient of x^{k-1} of $Q_{0,q}(x)$ is 1.

Remark

1. If we define $r_{k,q,l}$ by

$$r_{k,q,l} = \inf_{\delta} \inf \{r : R_{k,\delta}(x, q, l) \ll x^r\}$$

Theorem 2 shows that

$$\max_l r_{k,q,l} = \Theta_q.$$

The statement $\Theta_q = 1/2$ for every q is equivalent to the truth of the Riemann hypothesis for the Dirichlet L -function.

2. Similar results are hold for the following sums :

$$\omega_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ \omega(n)=k}} 1, \quad \Omega_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ \Omega(n)=k}} 1.$$

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